Ch. 3: Vectors and Two-Dimensional Motion

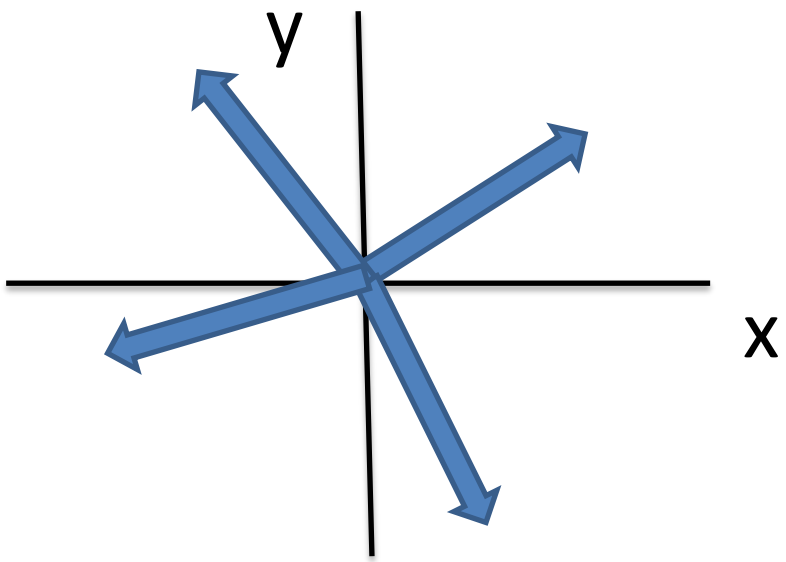
One Dimension

- So far: we have been dealing with motion in one dimension (along a single line)
- In one dimension, we can describe each quantity (position, velocity, acceleration, etc.) by a single number.
- Directions are given by a + or – sign.

v = + 5 m/s **v = + 5 m/s**

Two Dimensions

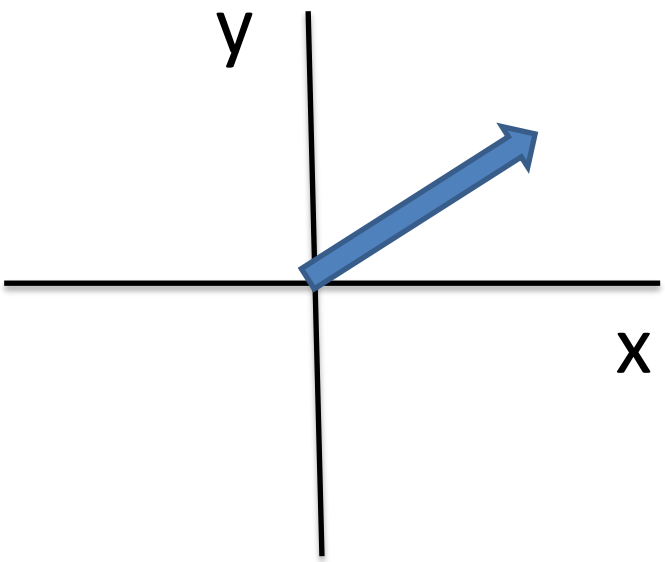
- In two dimensions, we have two coordinates (x = horizontal and y = vertical)
- There are now an **infinite number** of possible directions for movement.
- A single plus/minus sign will no longer cut it.



A few of the possible directions an object can move in two dimensions

Two Dimensions

- In **two dimensions**, quantities describing motion (velocity, acceleration, etc.) require **two numbers** to describe them.
- Similarly, in **three dimensions** each quantity is described by **three numbers**, etc.



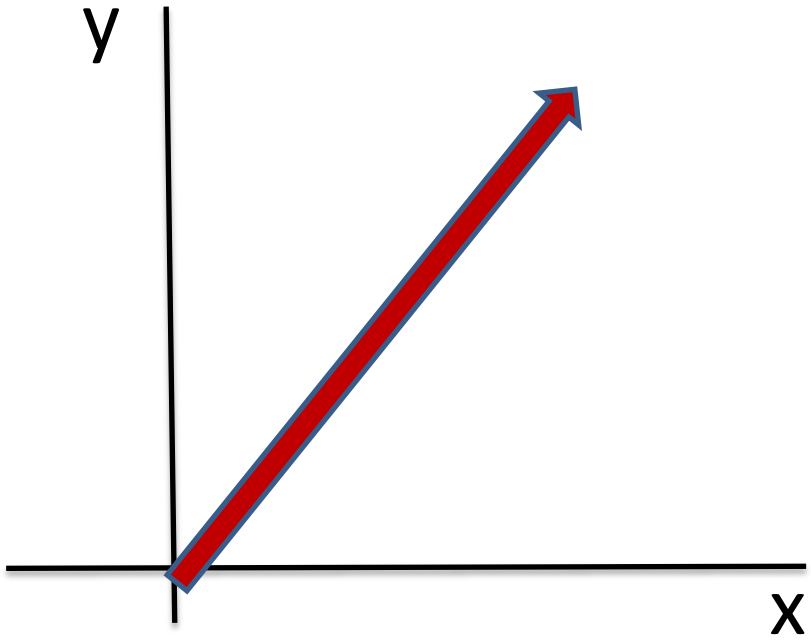
- This extra information is needed to describe the direction of motion.
- Quantities that encode information are called **vectors.**
- Quantities with no direction are called **scalars**.

Vectors versus scalars

- A **scalar** is an object that has just a size (magnitude). It is just a number
- A **vector** is an object with size **and** direction. Think of vectors as arrows.
- **Examples:**
  - **Scalars:** temperature, pressure, charge, energy
  - **Vectors:** velocity, acceleration, force, electric and magnetic fields.

Vectors (2D)

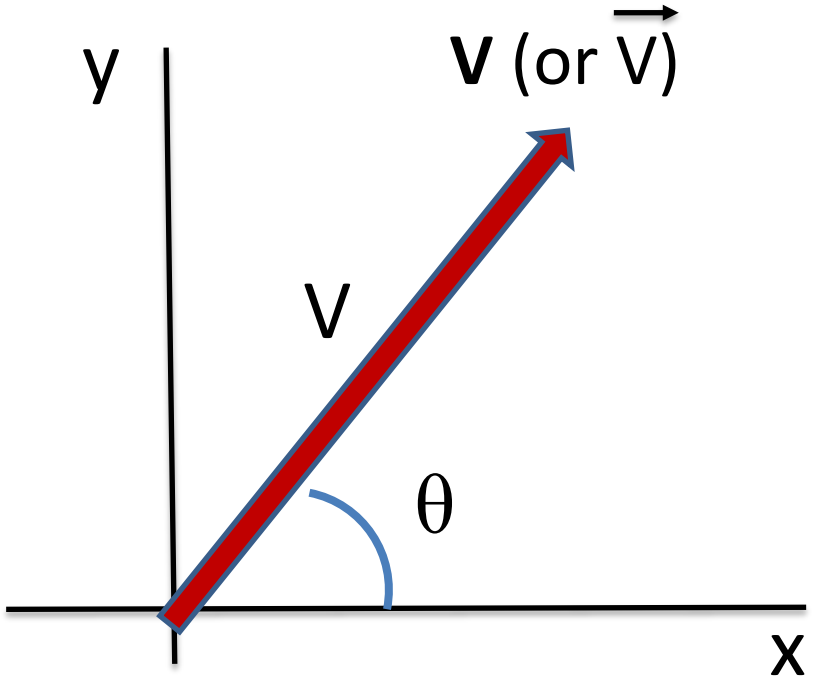
- Think of vectors as arrows. They carry two pieces of information:
  - **Magnitude or size** (length of the arrow)
  - **Direction** (angle of arrow from positive x-axis)



- These two numbers provide a complete description of a vector (in two dimensions)
- We'll soon see a second equivalent (and more useful) description.

Vectors: Notation

- Vectors are given names that are boldface or have arrows over them, to distinguish them from scalars. ($\mathbf{V}$, $\vec{V}$, $\mathbf{A}$, $\vec{A}$, $\boldsymbol{\alpha}$, $\vec{\alpha}$,……)



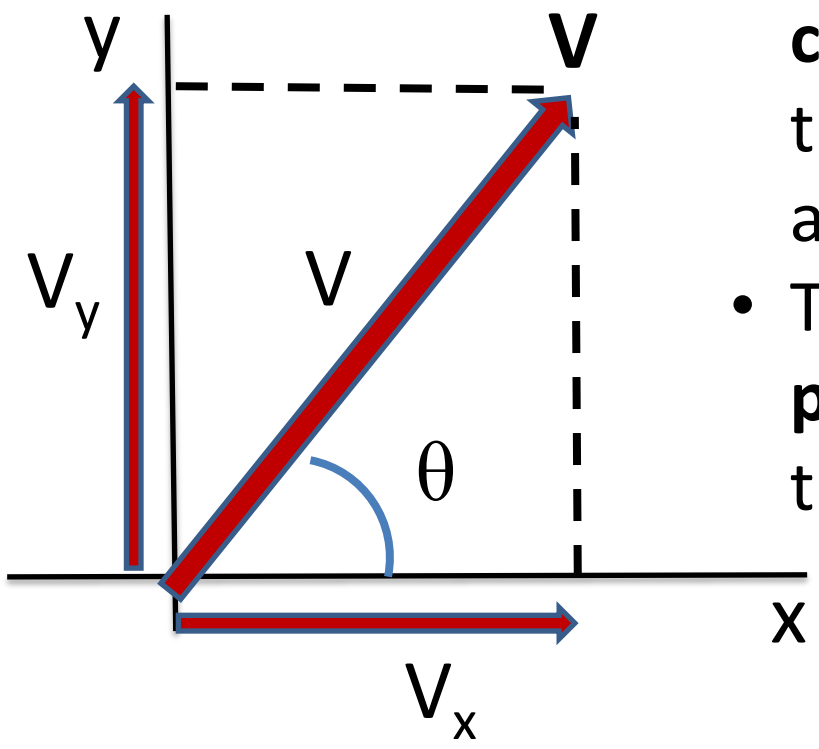
- The magnitude is denoted by the same name, but without the bold or the arrow.
- The angle is usually called θ or φ.

Theta Phi

- So one vector is equivalent to two numbers: **V**= (V, θ)
- (In n dimensions, one vector is equivalent to n numbers)

Vectors: Components

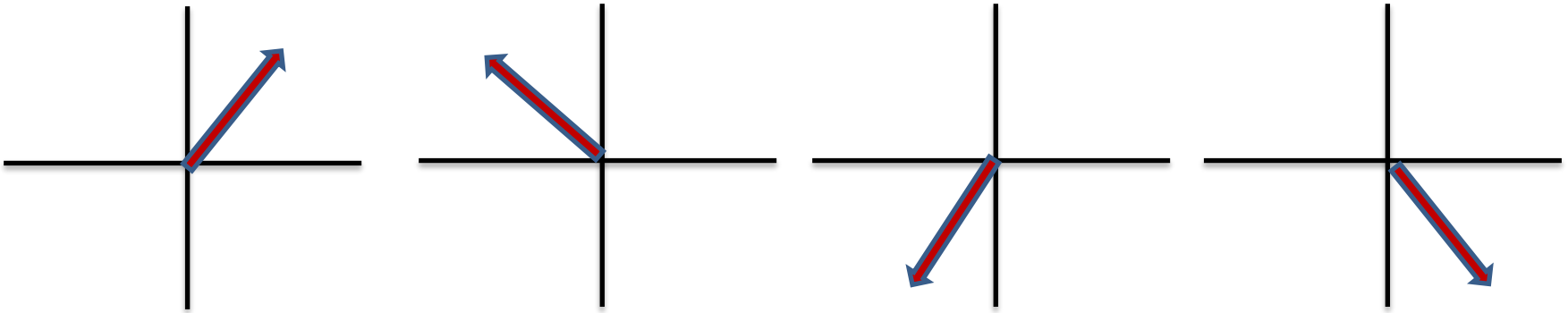
- There is a second way to describe a vector using two different numbers, called **components.**



- The **x-component** V_x and the **y-component** V_y are the distances the vector extends horizontally and vertically.
- They are the shadows or **projections** of the vector onto the two axes.

Signs

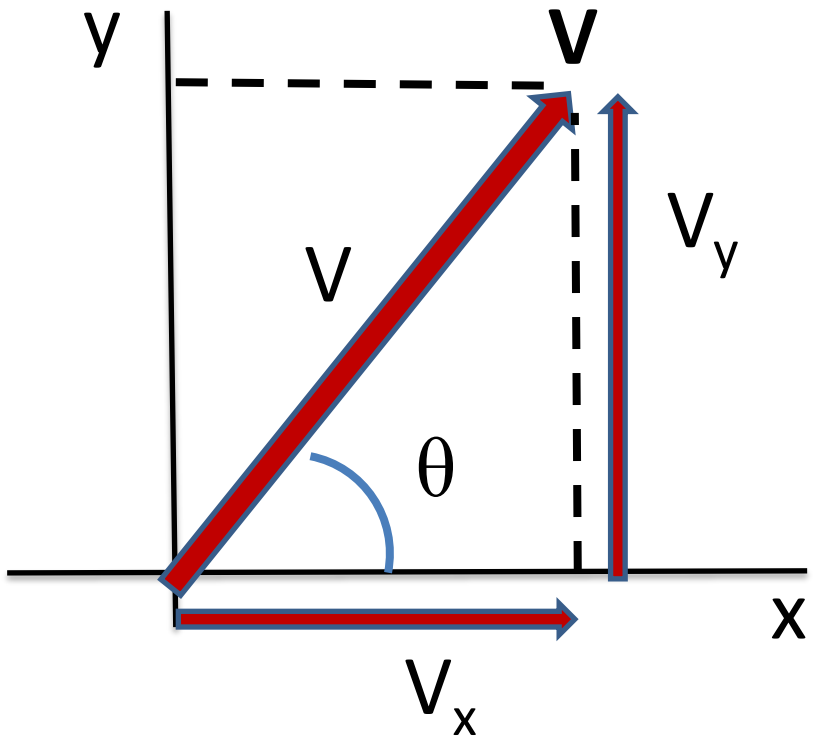
- Note that the **magnitude** of a vector **must always be positive**.
- But the **components** can be positive or negative, depending on direction.
- $V_x > 0 \rightarrow$ points right, $V_x < 0 \rightarrow$ points left
- $V_y > 0 \rightarrow$ points up, $V_y < 0 \rightarrow$ points down



$V_x>0$ $V_y>0$ | $V_x<0$ $V_y>0$ | $V_x<0$ $V_y<0$ | $V_x>0$ $V_y<0$

Converting between two descriptions

- We have two descriptions of the vector: in terms of magnitude and angle, and in terms of x- and y-components.



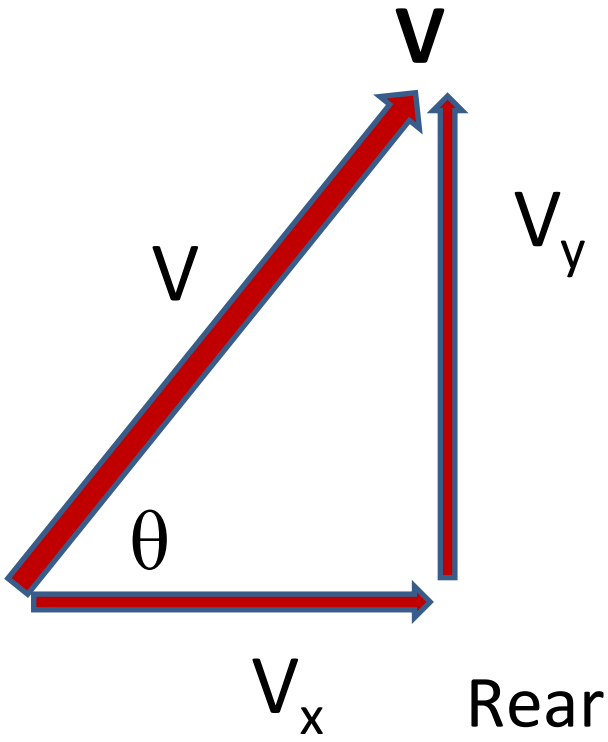
- To translate between them, notice that magnitude forms the hypotenuse of a right triangle.
- The two components form the two sides.

Converting between two descriptions

- From the Pythagorean theorem:

$$V^2 = V_x^2 + V_y^2$$

- From the definitions of the trig functions:



$$\sin\theta = \frac{opp}{hyp} = \frac{V_y}{V}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{V_x}{V}$$

$$\tan\theta = \frac{opp}{adj} = \frac{V_y}{V_x}$$

Rearrange these to get what we need

Converting between two descriptions

- Magnitude and angle from components:

$$V = \sqrt{V_x^2 + V_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{V_y}{V_x}$$

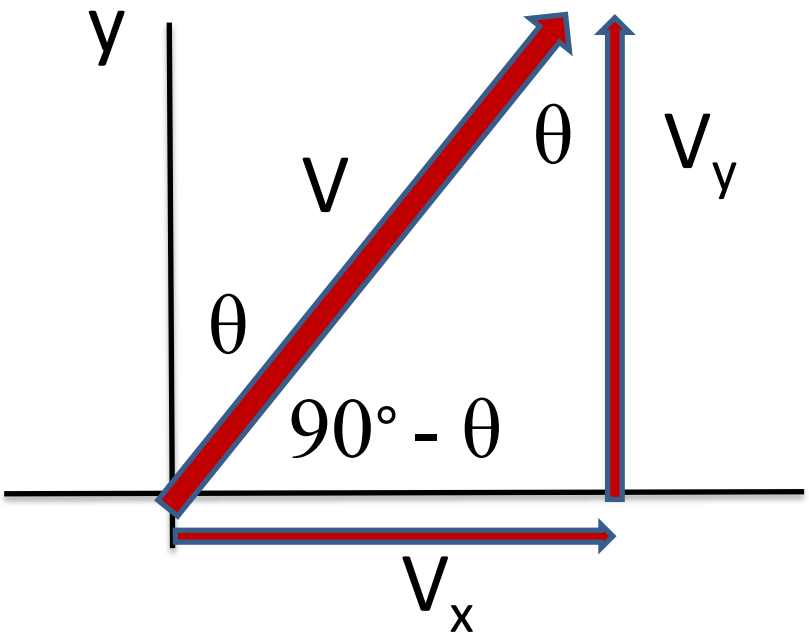
- From the definitions of the trig functions:

$$V_x = V\cos\theta \quad \text{and} \quad V_y = V\sin\theta$$

- These are the basic equations for working with vectors.

Converting between two descriptions

- **Be careful:** the previous equations assume is the angle from the **x-axis**.
- Often in problems you will be given the angle from the **y-axis**. In that case, sines and cosines get interchanged:

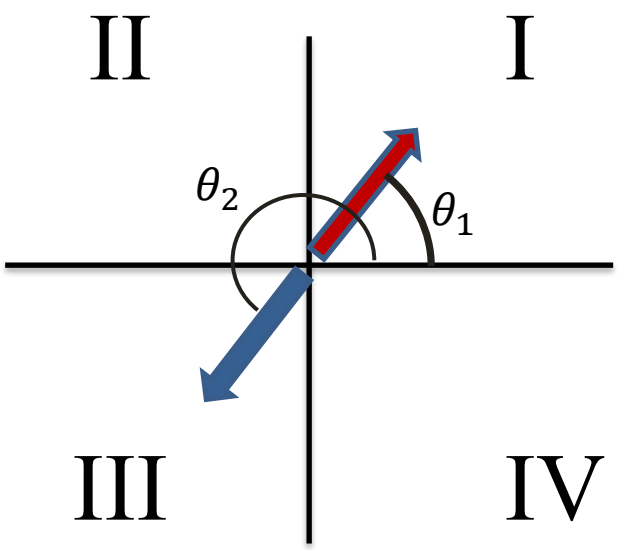


$$V_x = V\cos(90° - \theta)$$
$$= V\sin(\theta)$$

$$V_y = V\sin(90° - \theta)$$
$$= V\cos(\theta)$$

Converting between two descriptions

- **<u>One more warning</u>**: The inverse tangent is double-valued: check quadrant to make sure your calculator is giving the value you need.
- If it is in the wrong quadrant, add 180°.



The angles of the two vectors shown (180° apart, in quadrants I and III) have the same inverse tangent:

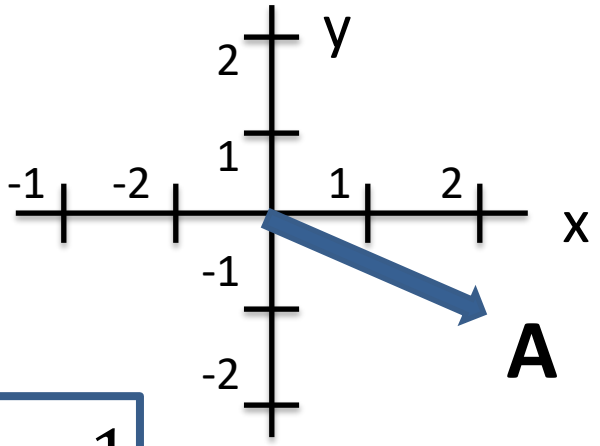
$$\tan\theta_1 = \frac{V_{1y}}{V_{1x}} = \frac{(-V_{2y})}{(-V_{2x})} = \tan\theta_2$$

$$(\theta_2 = \theta_1 + 180°)$$

Example 1

- A vector **A** is as shown.

(a) Find the components.



These can be read off directly from the picture:

$$A_x = +2, \quad A_y = -1$$

(b) Find the magnitude and direction

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(2)^2 + (-1)^2} = \sqrt{5} \approx 2.236$$

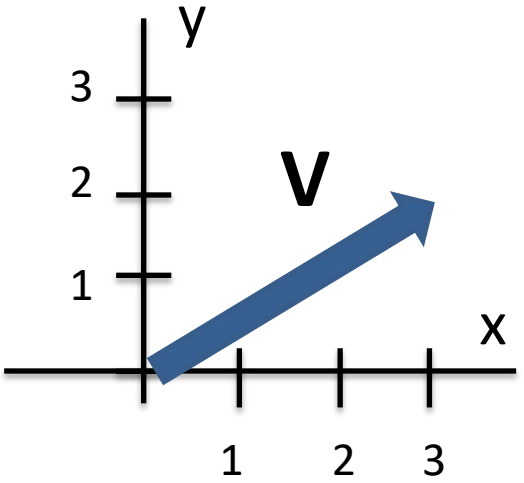
$$\theta = \tan^{-1}\frac{A_y}{A_x} = \tan^{-1}\left(\frac{-1}{2}\right) = -26.6° = 333.4°$$

$$= 5.82\, rad$$

Example 2

- A hiker walks three miles east, then 2 miles north. What is the magnitude and direction of the total displacement?

Given: $V_x = 3\ miles, V_y = 2\ miles$



$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{3^2 + 2^2}$$

$$= \sqrt{13} = \boxed{3.61\ miles}$$

$$\theta = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\frac{2}{3} = \boxed{33.69°}$$

$(= .59\ rad$, N of E)

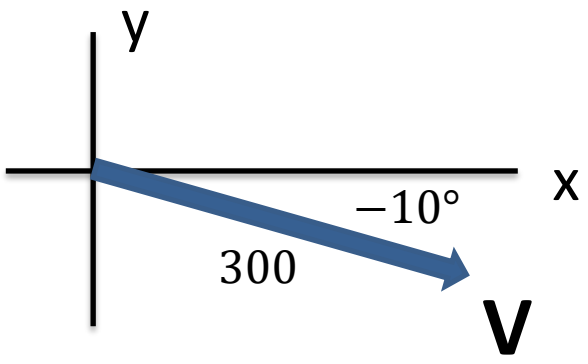
Example 3

- An airplane is moving at a constant speed of 300 mph, at $10°$ below horizontal. Find the components of the velocity.

Given: $V = 300\,mph, \quad \theta = -10°$

$$V_x = V\cos\theta = 300\ \cos(-10°) = \boxed{295.44\,mph}$$

$$V_y = V\sin\theta = 300\ \sin(-10°) = \boxed{-52.09\,mph}$$

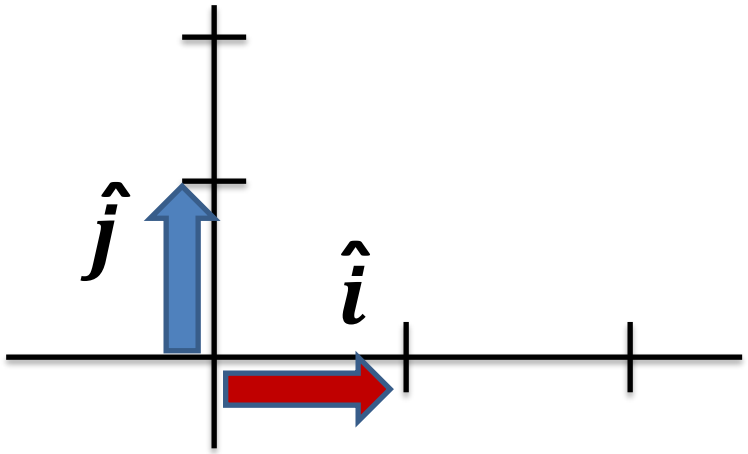


Check answer:

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{295.44^2 + (-52.09)^2} = 300\,mph$$

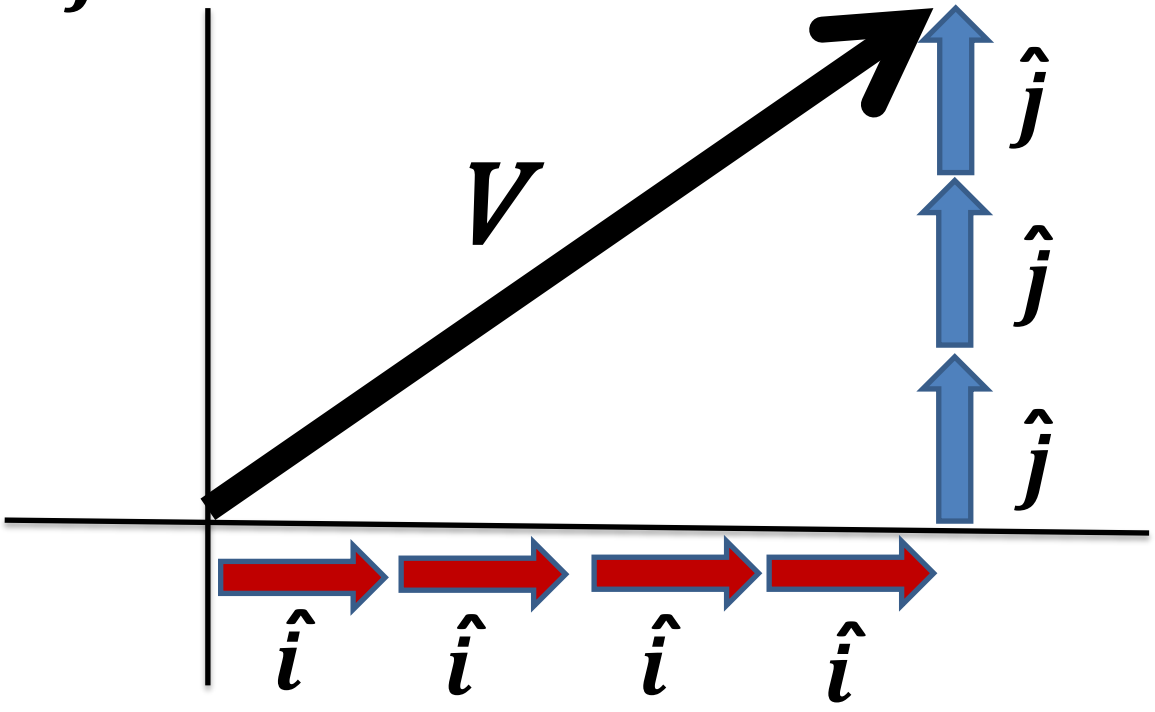
Unit Vector Notation

- When we start adding vectors, we will need a better notation for them.
- Define **unit vectors**: $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$
- These are vectors of length 1, pointing along the axes. The hats will always indicate unit vectors.



Unit Vector Notation

- Any vector can be built out of unit vectors.
- For example, the vector below can be written as $\boldsymbol{V} = 4\,\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$



Vectors in three dimensions

- In three dimensions, there are three coordinates (x, y, z) and three components for each vector ( )
- There are also now three unit vectors: $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{k}}$
- There will be **two** angles (one from the z-axis, one from the xy-plane).
- The Pythagorean theorem is now: $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$
- See the book for formulas for the components in terms of the two angles (we won't need them).

$$V = V_x\,\hat{i} + V_y\,\hat{j} + V_z\,\hat{k}$$

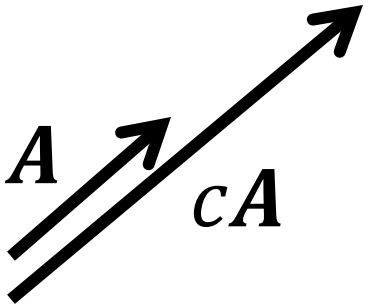
Operations on Vectors

- Given: Two vectors (***A*** and ***B***) and a positive number (c). What can we do with them?

1. <u>Multiplication of a vector by a positive number</u>: $\boldsymbol{A} \rightarrow c\boldsymbol{A}$.

This rescales length and components of $\boldsymbol{A}$ by a factor of c with no effect on direction:

$$A \rightarrow cA$$
$$A_x \rightarrow cAx$$
$$A_y \rightarrow cAy$$
$$\theta \rightarrow \theta$$



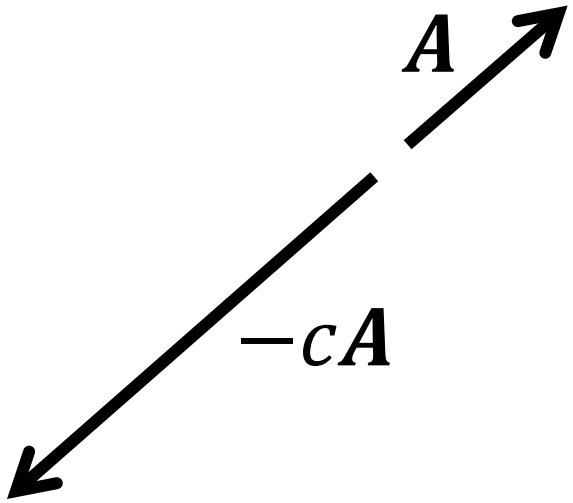
Operations on Vectors

2. Multiplication of a vector by a negative number:

$$\boldsymbol{A} \rightarrow -c\boldsymbol{A} \ .$$

This rescales length of $\boldsymbol{A}$ by a factor of c and components by $-c$. Direction is flipped by $180°$.

$$A \rightarrow cA$$
$$A_x \rightarrow -cA_x$$
$$A_y \rightarrow -cAy$$
$$\theta \rightarrow \theta + 180°$$

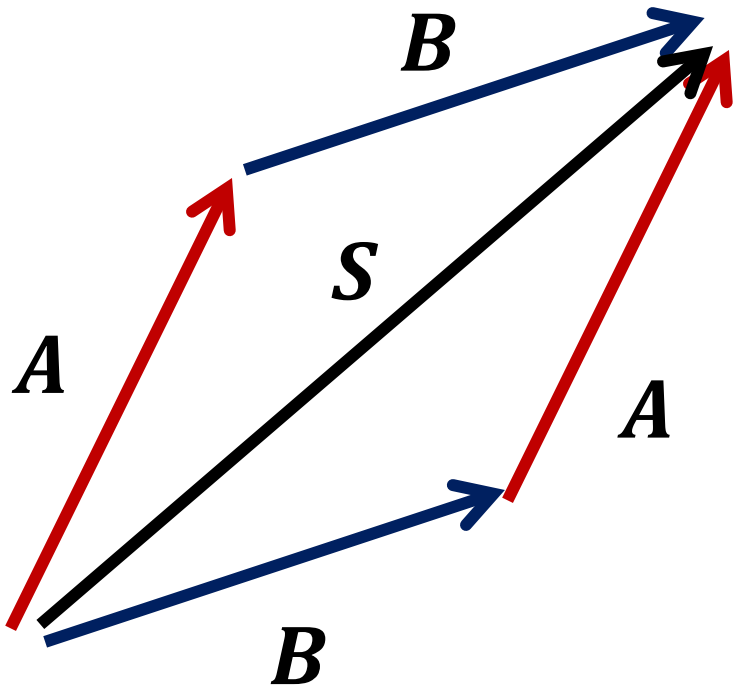


Operations on Vectors

3. <u>Addition of two vectors</u>: $\boldsymbol{S = A + B}$

Move one vector so its tail is at the tip of the other (parallelogram rule).

$$S_x = A_x + Bx$$
$$S_y = A_y + B_y$$



Note: you can only add components, you **CANNOT** add angles or magnitudes:

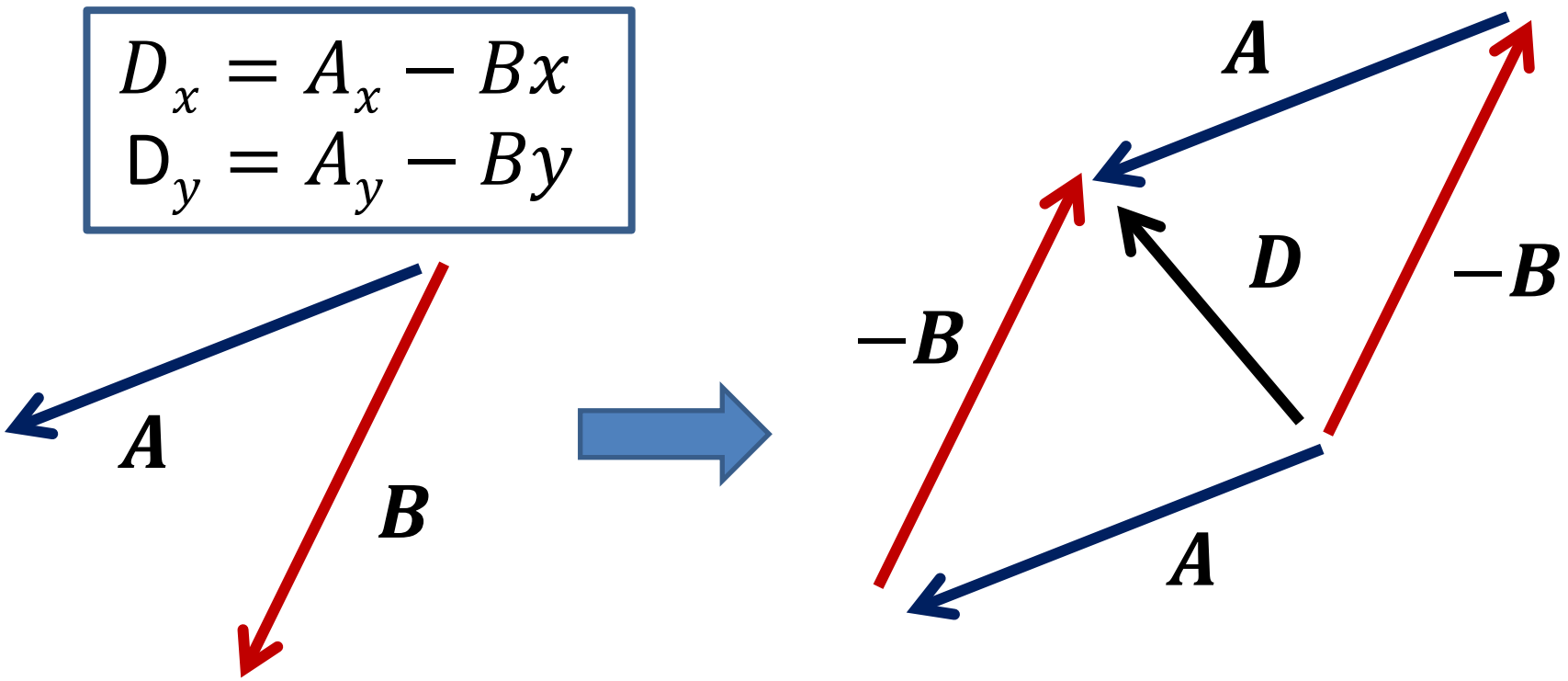
$$\theta_S \neq \theta_A + \theta_B$$
$$S \neq A + B$$

Operations on Vectors

3. Subtraction of two vectors: $\boldsymbol{D = A - B}$

Add the vector $-\boldsymbol{B}$ to the vector $\boldsymbol{A}$.

$$D_x = A_x - Bx$$
$$\mathrm{D}_y = A_y - By$$



Operations on Vectors

4. <u>Multiply vectors times vectors.</u>

   There are two types of product:

   - The cross product or vector product:

$$\boldsymbol{A} \times \boldsymbol{B} = a\ new\ vector$$

   - The dot product or scalar product:

$$\boldsymbol{A} \cdot \boldsymbol{B} = a\ scalar$$

We'll come back to these later.

Operations on Vectors

What you **CANNOT** do:

1. You **cannot** add or subtract scalars with vectors: $c \pm \boldsymbol{A}$
2. You **cannot** divide one vector into another: $\boldsymbol{A}/\boldsymbol{B}$

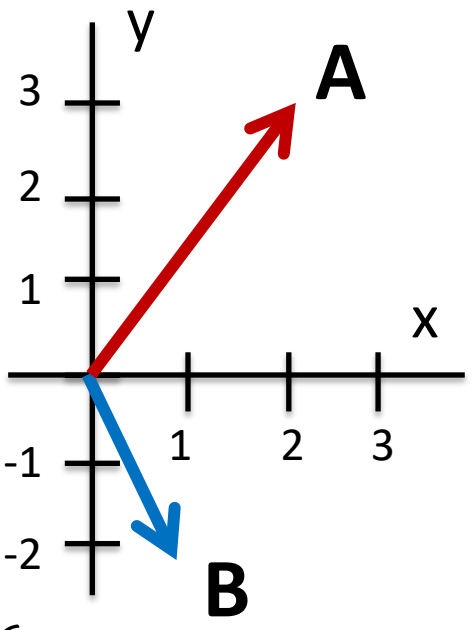
Once again: You also **cannot** add the magnitudes of vectors. To add vectors, you must break into components and **add the components**.

Example 1

- Two vectors A and B are given, as shown.

(a) Find the magnitude and direction of each vector.

$$A_x = 2,\ A_y = 3,\ B_x = 1,\ B_y = -2$$



For **A**:

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{2^2 + 3^2} = \boxed{\sqrt{13}} \approx 3.61$$

$$\theta_A = \tan^{-1}\frac{A_y}{A_x} = \tan^{-1}\frac{3}{2} = \boxed{56.31°}$$

For **B**:

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{1^2 + (-2)^2} = \boxed{\sqrt{5}} \approx 2.236$$

$$\theta_B = \tan^{-1}\frac{B_y}{B_x} = \tan^{-1}\left(\frac{-2}{1}\right) = \boxed{-63.4°}$$

Continued

Example 1 (cont.)

(b) Sketch $\boldsymbol{C} = -3\boldsymbol{B}$ and find its components.

Components:

$$C_x = -3B_x = -3(1) = \boxed{-3}$$

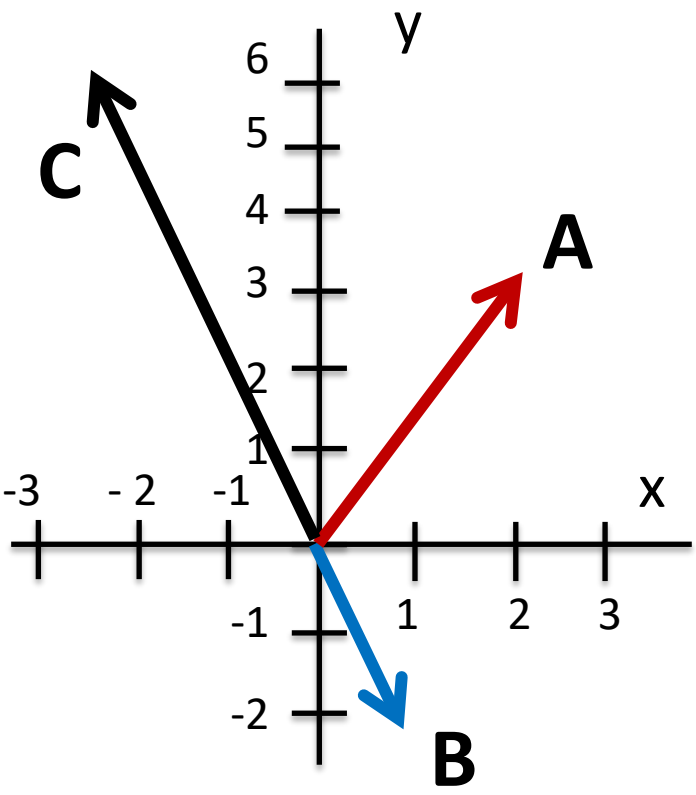
$$C_y = -3B_y = -3(-2) = \boxed{+6}$$

Also:

$$C = 3|B| = 3\sqrt{5}$$

$$\theta_C = \theta_B + 180°$$

$$= -63.4° + 180° = 117.6°$$



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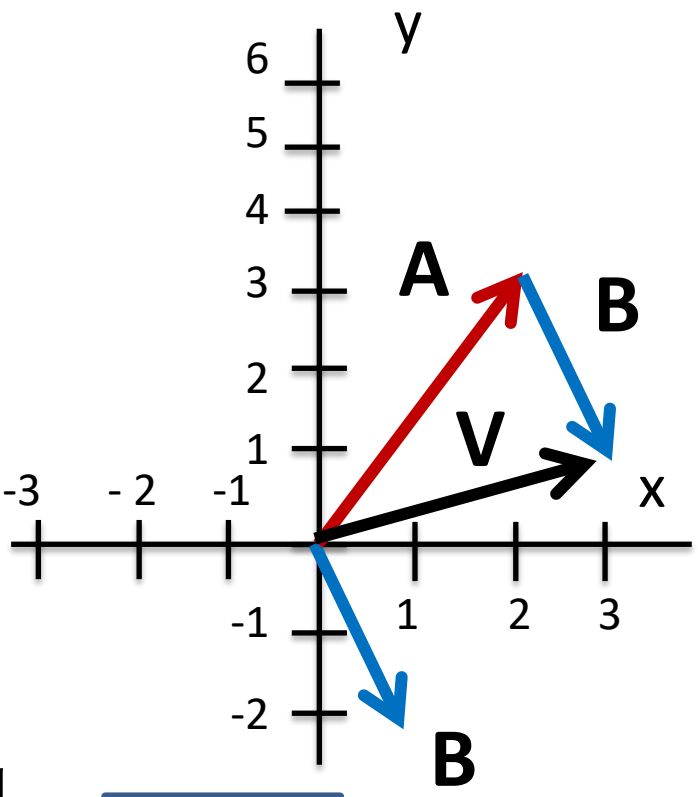
Example 1 (cont.)

(b) Find magnitude and direction of $\boldsymbol{V} = \boldsymbol{A} + \boldsymbol{B}$.



Components:

$$V_x = A_x + B_x = 2 + 1 = 3$$

$$V_y = A_y + B_y = 3 + (-2) = 1$$

Magnitude & direction:

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{3^2 + 1^2}$$

$$= \boxed{\sqrt{10}} \approx 3.16$$

$$\theta_V = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\frac{1}{3} = \boxed{18.43°}$$

Continued

Example 1 (cont.)

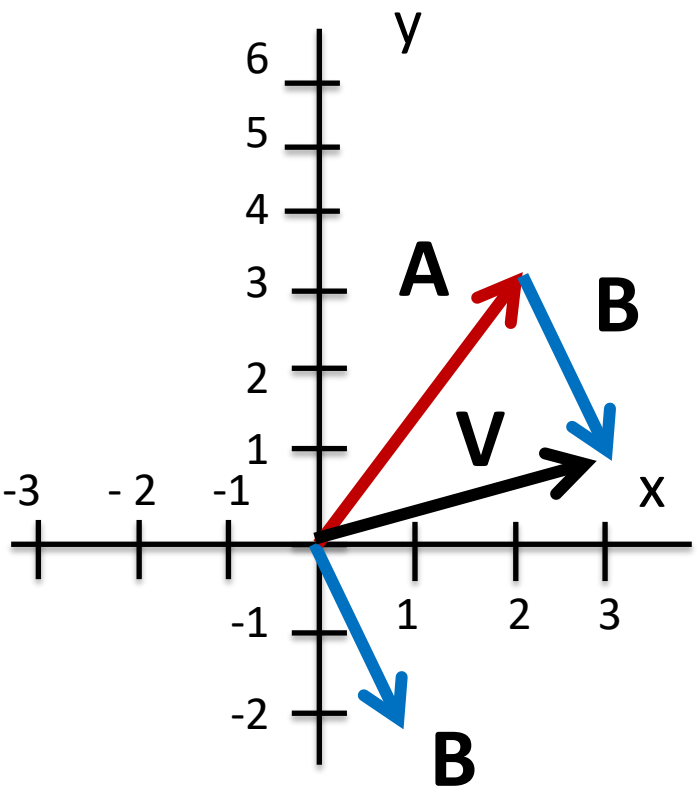
Again - Notice that angles and lengths cannot be added:

$$\theta_V \neq \theta_A + \theta_B$$

$18.43°$ $\quad -63.4°$ $\quad 56.31°$

$$V \neq A + B$$

3.16 $\quad 3.61$ $\quad 2.236$

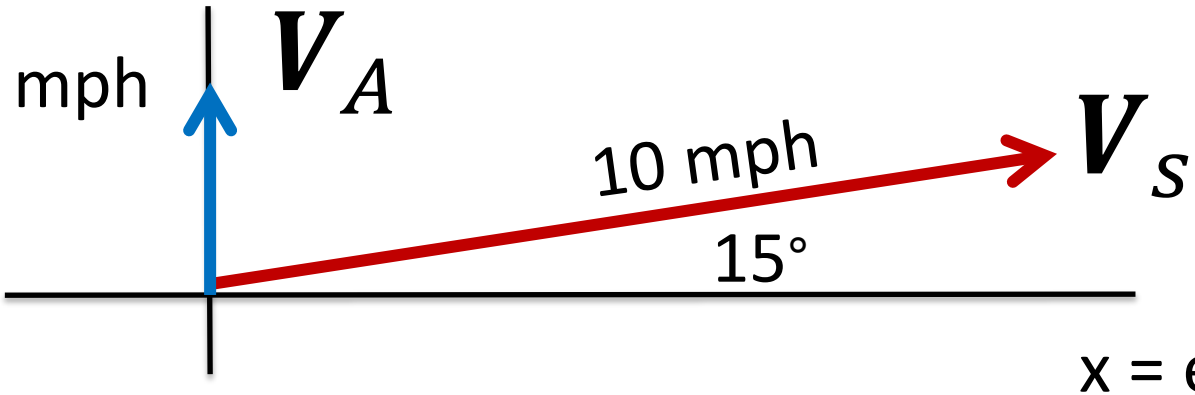


Example 2

- A sailboat is traveling at 10 mph, 15° N of E. A gust of wind gives it an additional velocity of 2 mph due north.
- What is the total final velocity?

y = north

Total: $\boldsymbol{V} = \boldsymbol{V}_S + \boldsymbol{V}_A$



Example 2 (continued)

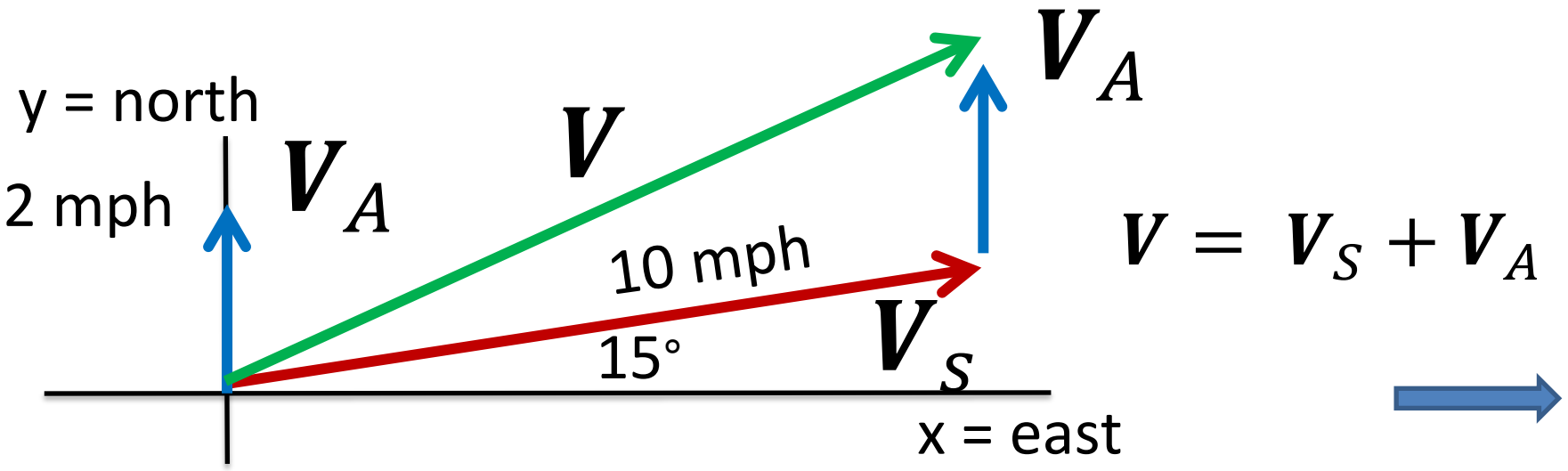
- To add the vectors, we first need components:

$$V_{Sx} = V_S \cos 15° = 10 \cos 15° = 9.66\ mph$$

$$V_{Sy} = V_S \sin 15° = 10 \sin 15° = 2.59\ mph$$

$$V_{Ax} = V_A \cos 90° = 2 \cos 90° = 2\ mph$$

$$V_{Ay} = V_A \sin 90° = 2 \sin 90° = 0$$



Example 2 (continued)

- Total: $V_x = V_{Sx} + V_{Ax} = 9.66 + 0 = 9.66\ mph$

  $V_y = V_{Sy} + V_{Ay} = 2.59 + 2 = 4.59\ mph$

- Magnitude and direction:

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{9.66^2 + 4.59^2} = \boxed{10.7\ mph}$$

$$\theta_V = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\frac{4.59}{9.66} = \boxed{25.41°}$$

Products of Vectors

- There are two completely different types of products that can be defined for vectors.
- Suppose we are given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Then we can form:
  - The **dot product** or **scalar product**: $\boldsymbol{A} \cdot \boldsymbol{B}$
  - The **cross product** or **vector product**: $\boldsymbol{A} \times \boldsymbol{B}$
- **<u>Examples</u>** of each in physics:
  - **Dot products:** work, electric force on dipole
  - **Cross product:** torque, magnetic forces

Dot Products

- Given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, we can construct a product called the **dot product** or **scalar product**: $\boldsymbol{A} \cdot \boldsymbol{B}$
- The result of this product is a **scalar**: it has no components or direction.
- The order of the multiplication doesn't matter: $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$
- The dot product can be positive or negative (or zero)

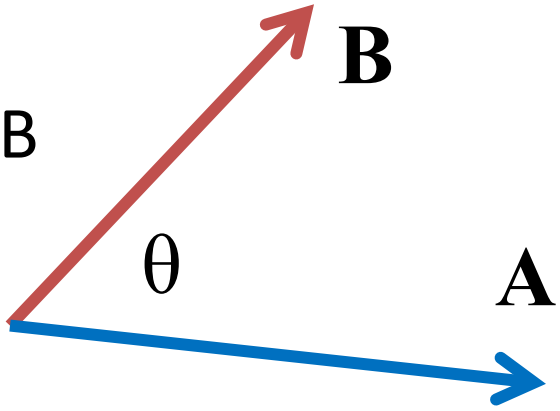
Dot Products

- There are two equivalent formulas for the dot product:

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y$$

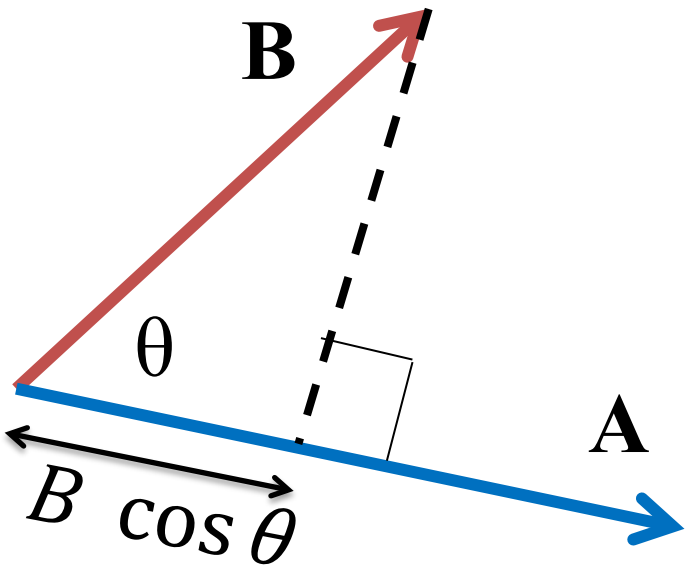
$$\boldsymbol{A} \cdot \boldsymbol{B} = AB \cos\theta$$

- θ is the angle from between **A** and **B**, **not** the angle from the x-axis.
- In the second equation, the factor AB on the right is the product of the **magnitudes**.
- In three dimensions, the first equation becomes: $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$



Dot Products

- The formula $\boldsymbol{A} \cdot \boldsymbol{B} = AB \cos\theta$ has a simple geometric meaning.
- $\boldsymbol{B} \cos \boldsymbol{\theta}$ is the **projection** or **component** of ***B*** in the direction of ***A***
- The dot product multiplies the parts of A and B that are parallel to each other.
- The perpendicular parts don't contribute to the product.



Example 1

- Suppose two vectors of lengths $A = 3$ and $B = 4$ are separated from each other by $25°$.
- Find $\boldsymbol{A} \cdot \boldsymbol{B}$.

We are given lengths and the angle between the vectors, so use second formula.

$$\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{AB}\cos\theta = (3)(4)\cos 25° = \boxed{10.9}$$

Example 2

- Given two vectors: $\boldsymbol{A} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}, \quad \boldsymbol{B} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

(a) Find $\boldsymbol{A} \cdot \boldsymbol{B}$. $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$

$$= (1)(2) + (-1)(-3) + (1)(2) = \boxed{7}$$

(b) Find the angle between **A** and **B**.

$$\mathrm{A} = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\mathrm{B} = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{2^2 + (-3)^2 + 2^2} = \sqrt{17}$$

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB\ \cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{AB} = \cos^{-1}\frac{\boldsymbol{7}}{\sqrt{3}\sqrt{17}}$$

$$= \boxed{11.47°}$$

Cross product

- There is a second type of product for vectors: the cross product or vector product, $\boldsymbol{A} \times \boldsymbol{B}$.
- Unlike the dot product the cross product results in a new vector.
- It is also **non-commutative** – order matters:

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$$

Cross product

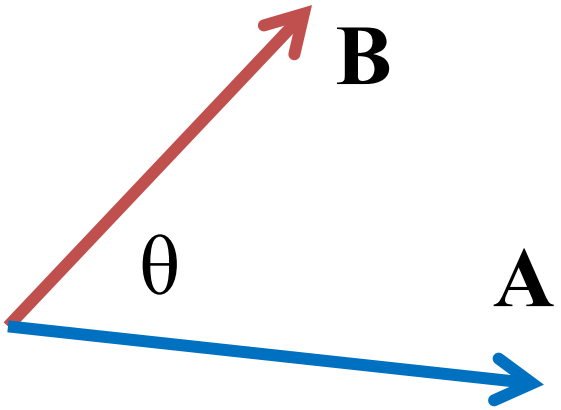
- Definition of cross product:

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}}$$

- Magnitude of cross product:

$$|\boldsymbol{A} \times \boldsymbol{B}| = AB \sin\theta$$

(where θ is angle between **A** and **B**)



Cross product

- Magnitude of cross product:

$$|\boldsymbol{A} \times \boldsymbol{B}| = AB \cos\theta$$

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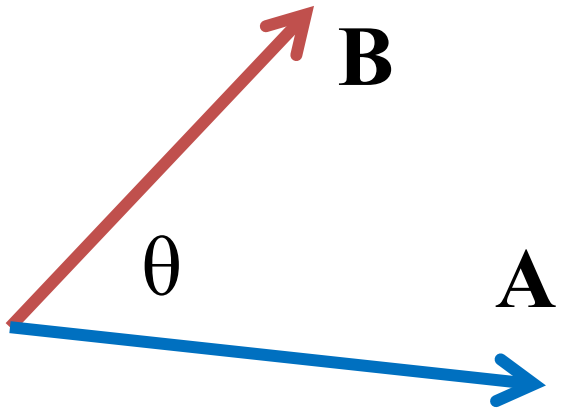
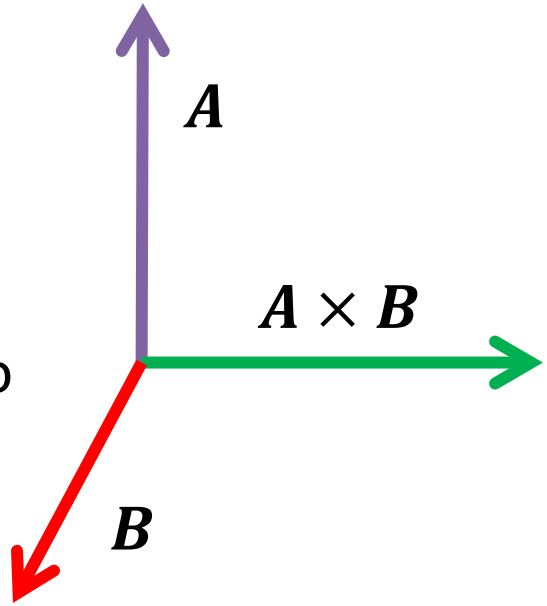
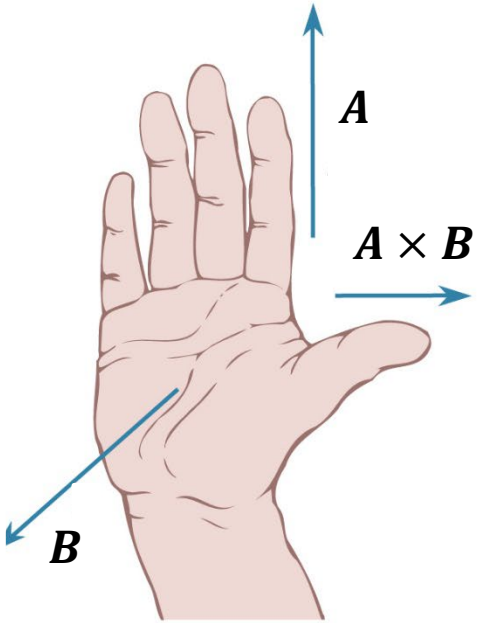


- Compare to dot product: $\boldsymbol{A} \cdot \boldsymbol{B} = AB \sin\theta$
- The dot product multiplies the parts of A and B that are parallel to each other. The cross product multiplies the perpendicular parts.
- Note: $\boldsymbol{A} \times \boldsymbol{B} = 0$ if A and B are parallel ($\theta = 0$)

Cross product: right-hand rule

- The direction of $\boldsymbol{A} \times \boldsymbol{B}$ is always perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$, and is given by the **right-hand rule.**
- Using **right** hand:
  - First vector ($\boldsymbol{A}$) along fingers
  - Palm in direction of 2nd vector ($\boldsymbol{B}$)
  - $\boldsymbol{A} \times \boldsymbol{B}$ is along thumb ($\perp$ to both $\boldsymbol{A}$ and $\boldsymbol{B}$)

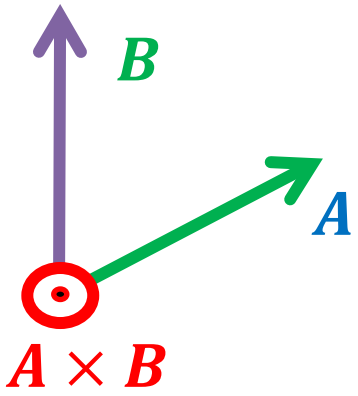


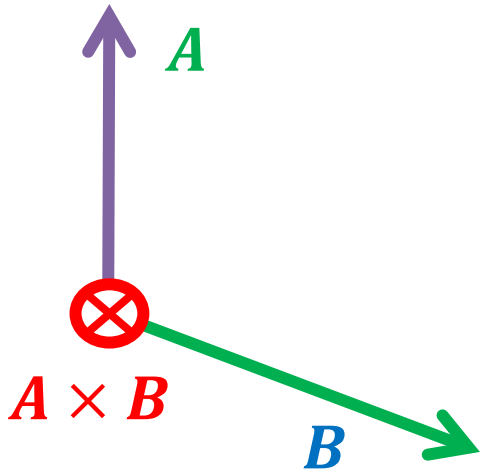


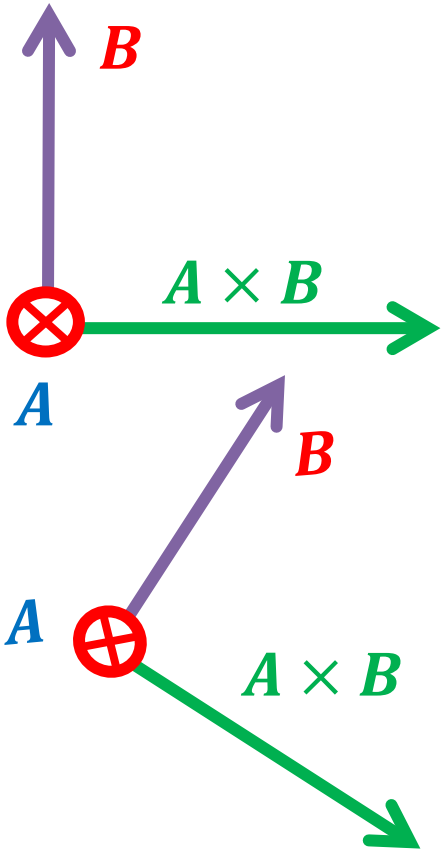
Cross product: right-hand rule

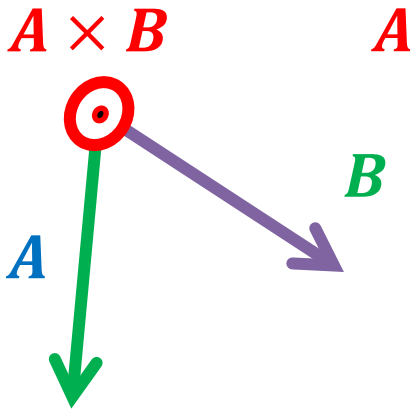
- Notation: ⊙ = pointing into page (or screen, etc)

  ⊗ = pointing out of page (or screen, etc)









Example

- Suppose $\boldsymbol{A} = 3\boldsymbol{i} - 4\boldsymbol{j}$ and $\boldsymbol{B} = -2\boldsymbol{i} + 3\boldsymbol{k}$.

$$\boldsymbol{A} \times \boldsymbol{B} = \left(A_y B_z - A_z B_y\right)\mathbf{i} + \left(A_z B_x - A_x B_z\right)\mathbf{j} + \left(A_x B_y - A_y B_x\right)\mathbf{k}$$

$$= [(-4)(3) - (-4)(0)]\,\mathbf{i} + [(0)(-2) - (3)(3)]\,\mathbf{j} + [(3)(0) - (-4)(-2)]\,\mathbf{k}$$

$$= \boxed{-12\,i - 9\,j - 8\,k}$$

Magnitude: $$|\boldsymbol{A} \times \boldsymbol{B}| = \sqrt{(\boldsymbol{A} \times \boldsymbol{B})_x^2 + (\boldsymbol{A} \times \boldsymbol{B})_y^2 + (\boldsymbol{A} \times \boldsymbol{B})_z^2}$$

$$= \sqrt{(-12)^2 + (-9)^2 + (-8)^2} = \boxed{17}$$

Continued

Example (continued)

- What is the angle between $\boldsymbol{A} = 3\boldsymbol{i} - 4\boldsymbol{j}$ and $\boldsymbol{B} = -2\boldsymbol{i} + 3\boldsymbol{k}$?

We know that: $|\boldsymbol{A} \times \boldsymbol{B}| = AB \sin\theta$

So:

$$\theta = \sin^{-1}\left(\frac{|\boldsymbol{A} \times \boldsymbol{B}|}{A\,B}\right) = \sin^{-1}\left(\frac{\mathbf{17}}{5\sqrt{13}}\right) = \boxed{70.56°}$$

$$|A| = A = \sqrt{A_x^2 + A_y^2} = \sqrt{3^2 + 4^2} = 5$$

$$|B| = B = \sqrt{B_x^2 + B_y^2} = \sqrt{2^2 + 3^2} = \sqrt{17}$$

Two dimensional motion

- In two dimensions, position and velocity are vectors and each have two components.

$$Position: \quad \boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$$

$$Velocity: \quad \boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j}$$

- Similarly, acceleration due to gravity is a vector: $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j},$

where:

$$a_x = 0,$$

$$a_y = -g = -9.8\ \frac{m}{s^2}$$

Projectile Motion

- Consider an object shot into the air, with initial speed v_0, at angle θ from the horizontal.
- What are the position & velocity as a function of time?
- Only acceleration is that due to gravity.
- Position and velocity are both vectors:

Initial	Final
$\boldsymbol{r_0} = x_0\boldsymbol{i} + y_0\boldsymbol{j}$	$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$
$\boldsymbol{v_0} = v_{x0}\boldsymbol{i} + v_{y0}\boldsymbol{j}$	$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j},$

where: $v_0 = \sqrt{v_{x0}^2 + v_{y0}^2}$

Projectile Motion

- Both components of the acceleration are constant.
- To describe projectile motion we simply use two copies of the constant acceleration equations:
  - One for the horizontal direction (with $a_x = 0$)
  - One for the vertical direction (with $a_y = -g$)

Projectile Motion

- General equations:

1. $x = x_0 + v_0 t + \frac{1}{2} a t^2$
2. $v = v_0 + at$
3. $v^2 = v_0^2 + 2a(x - x_0)$

$a_x = 0$ $\quad$ $a_y = -g$

Horizontal (x)	Vertical (y)
1. $x = x_0 + v_{x0} t$	1. $y = y_0 + v_{y0} t - \frac{1}{2} g t^2$
2. $v_x = v_{x0}$	2. $v_y = v_{y0} - gt$
3. $v_x^2 = v_{x0}^2$	3. $v_y^2 = v_{y0}^2 - 2g(y - y_0)$

Projectile Motion

- General equations:

1. $x = x_0 + v_0 t + \frac{1}{2} a t^2$
2. $v = v_0 + at$
3. $v^2 = v_0^2 + 2a(x - x_0)$

$a_x = 0$ $\quad$ $a_y = -g$

Horizontal (x)	Vertical (y)
1. $x = x_0 + v_{x0} t$	1. $y = y_0 + v_{y0} t - \frac{1}{2} g t^2$
2. $v_x = v_{x0}$	2. $v_y = v_{y0} - gt$
3. ~~$v_x^2 = v_{x0}^2$~~	3. $v_y^2 = v_{y0}^2 - 2g(y - y_0)$

Notice this equation is redundant.

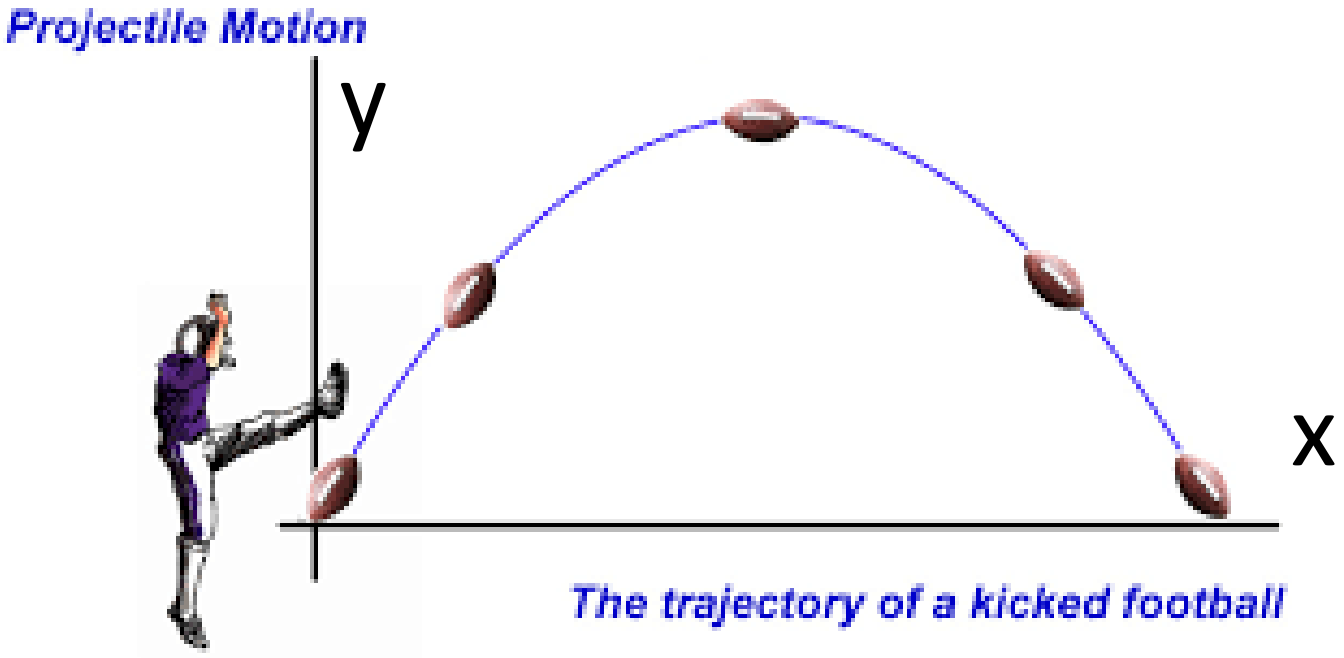
Projectile Motion

- So our main equations are:

Horizontal (x)	Vertical (y)
1. $x = x_0 + v_{x0}t$	1. $y = y_0 + v_{y0}t - \frac{1}{2}gt^2$
2. $v_x = v_{x0}$	2. $v_y = v_{y0} - gt$
	3. $v_y^2 = v_{y0}^2 - 2g(y - y_0)$

The motion of a projectile is parabolic

Projectile: Trajectory is a Parabola

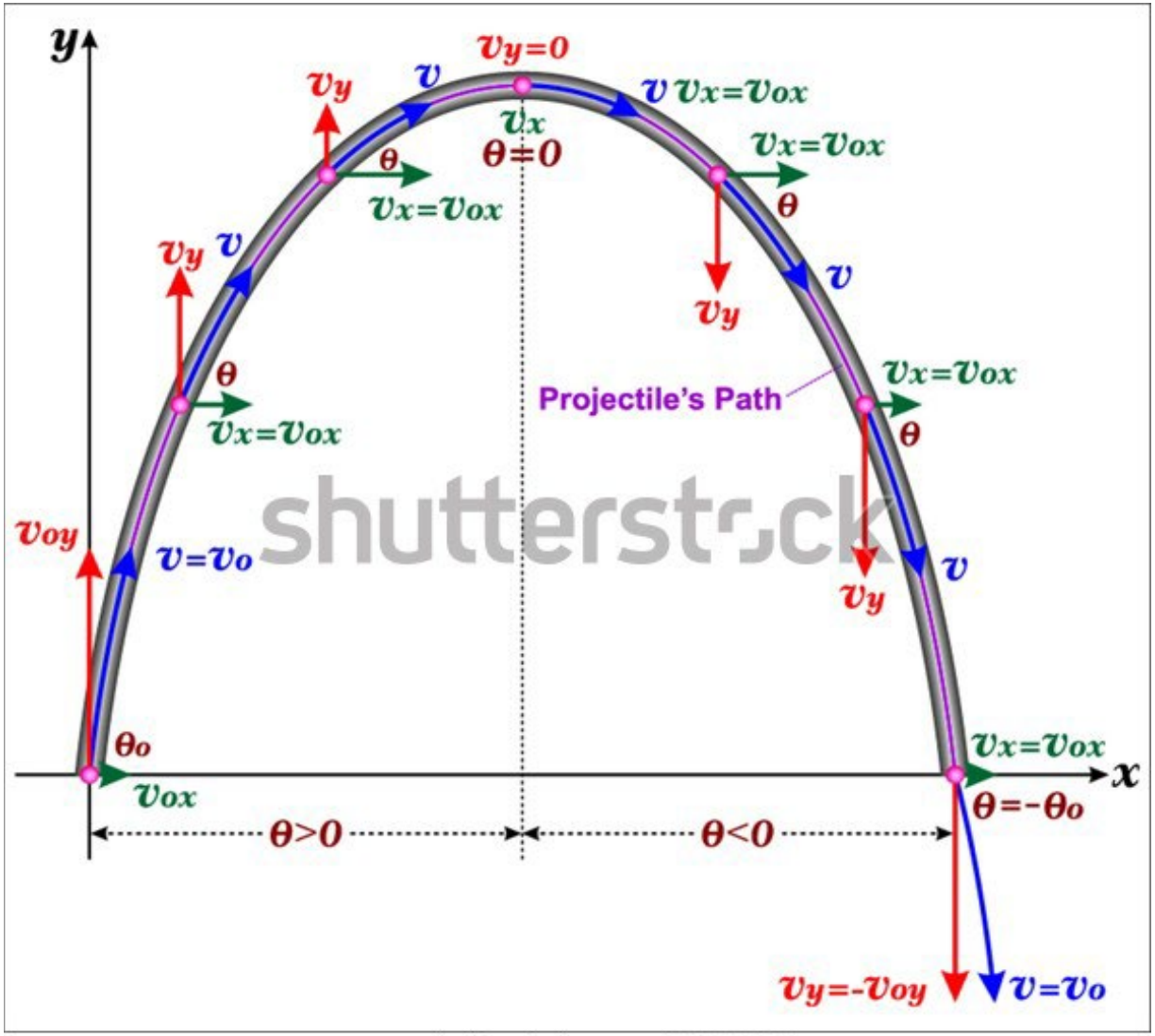


Plugging 2nd x equation into 2nd y equation to eliminate time, we find equation of parabola:

$$y = (\tan\theta)x - \left(\frac{g}{2v_0^2 cos^2\theta}\right)x^2$$

(General equation for parabola: $y = a x^2 + b x + c$)

Anatomy of Parabolic Motion



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- v starts positive, decreases over time, eventually becoming negative
- a is **always** negative (it is constant)
- At maximum height, $v_y = 0$.
- v_x is **not** zero at maximum height (remember it is constant), and **speed** is not zero.

Example 1

- A stuntperson drives motorcycle off cliff with initially horizontal velocity of 5 m/s. Find position and velocity after .25 s.
- Given: $v_{0x} = 5\frac{m}{s}, v_{0y} = 0,\ t = .25s$
- Define: $x_0 = 0, y_0 = 0$
- Want: $x,\ y, v_x, v_y$ at $t = .25s$

Example 1 (continued)

- Find position first:

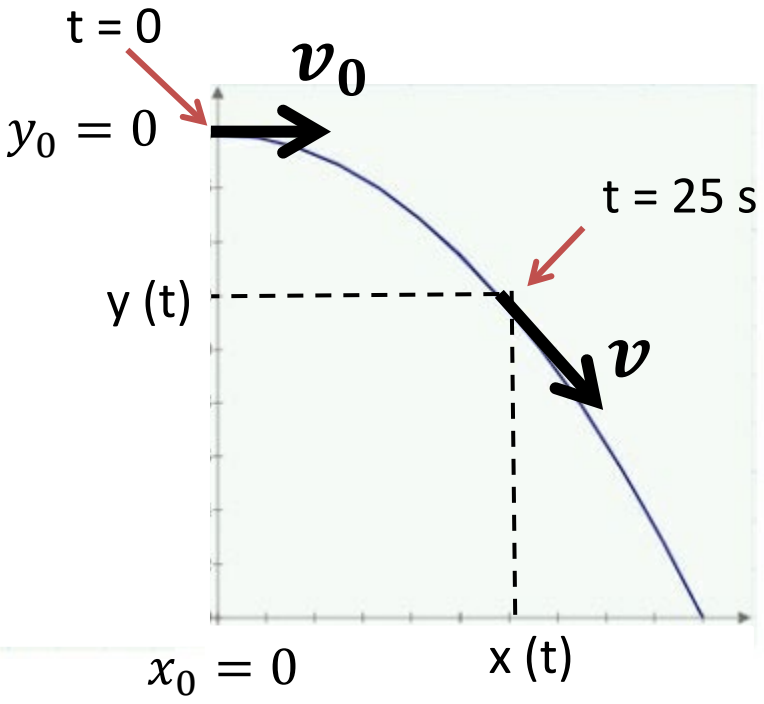
$$x = x_0 + v_{ox}t = 0 + \left(5\ \frac{m}{s}\right)(.25\ s)$$

$$= \boxed{1.25\ m}$$

$$y = y_0 + v_{oy}t - \frac{1}{2}gt^2$$

$$= 0 + 0 - \tfrac{1}{2}\left(9.8\tfrac{m}{s^2}\right)(.25\ s)^2$$

$$= \boxed{-.306\ m}$$



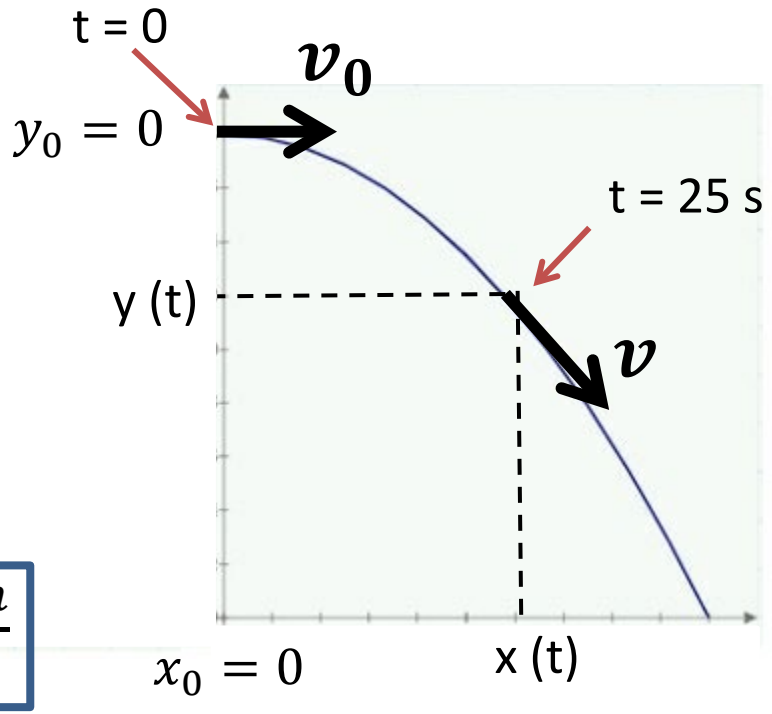
- Now find velocity

Example 1 (continued)

- Find velocity:

$$v_x = v_{0x} = \boxed{5\ \frac{m}{s}}$$

(since $a_x = 0$)



$$v_y = v_{0y} - gt$$

$$= 0 - \left(9.8\frac{m}{s^2}\right)(.25\ s) = \boxed{-2.45\frac{m}{s}}$$

- Speed: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{(5)^2 + (2.45)^2} = 5.57\ \frac{m}{s}$

- Direction: $\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\left(\frac{-2.45}{5}\right) = -26.1°$

Example 2

- A ball is thrown into the air at $40°$ above the horizon with initial speed $v_0 = 3\ {}^{m}/_{s}$.

(a) What is the speed when the ball is .1 m above initial height ($y - y_0 = .1\ m$).

First find components at t=0:

$$v_{0x} = v_0 \cos\theta = 3\ \cos 40° = 2.30\ {}^{m}/_{s}$$

$$v_{0y} = v_0 \sin\theta = 3\ \sin 40° = 1.93\ {}^{m}/_{s}$$

Find final components:

(v_y= 1.33 m/s)

$$v_x^2 = v_{0x}^2 = (2.3\ {}^{m}/_{s})^2 = 5.29\ {}^{m^2}/_{s^2}$$

$$v_y^2 = v_{0y}^2 - 2g(y - y_0) = (1.93)^2 - 2(9.8)(.1) = 1.76\ {}^{m^2}/_{s^2}$$

Find final speed:

$$v = \sqrt{v_x^2 + v_x^2} = \sqrt{5.29 + 1.76} = \boxed{2.66\ \text{m/s}}$$

Example 2 (continued)

(b) Find the maximum height the ball reaches (as measured from initial height).

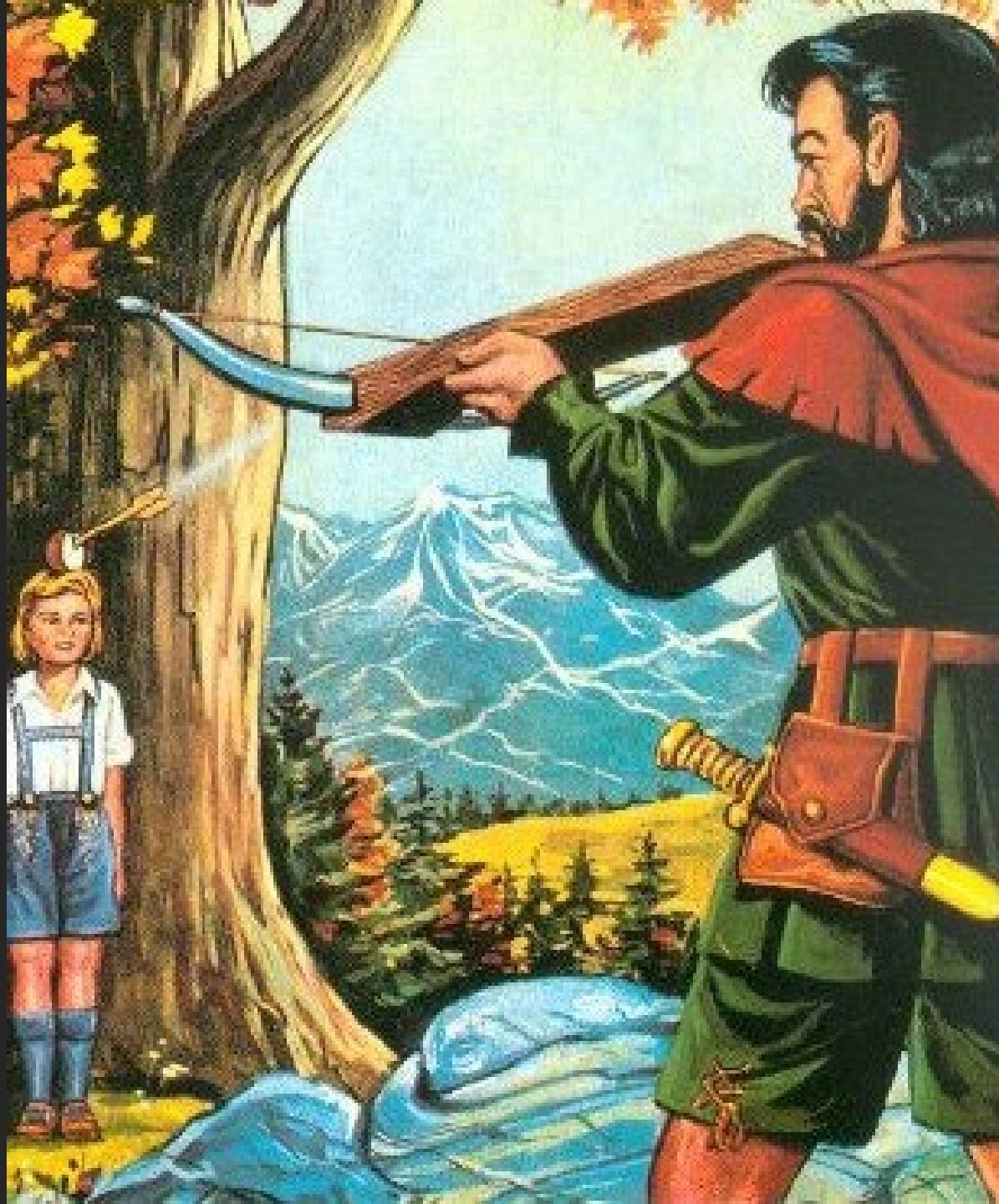
At maximum height, we know $v_y = 0$. So:

$$v_y^2 = v_{0y}^2 - 2g(y - y_0) \quad \rightarrow \quad 0 = v_{0y}^2 - 2gy$$

$$y = \frac{v_{0y}^2}{2g} = \frac{(1.93)^2}{(2)(9.8)} = \boxed{0.19\ m}$$

Note again that only v_y is zero at the maximum height, **not** v_x or the speed. Also, in part a, be careful not to confuse the components of the initial and final velocities.

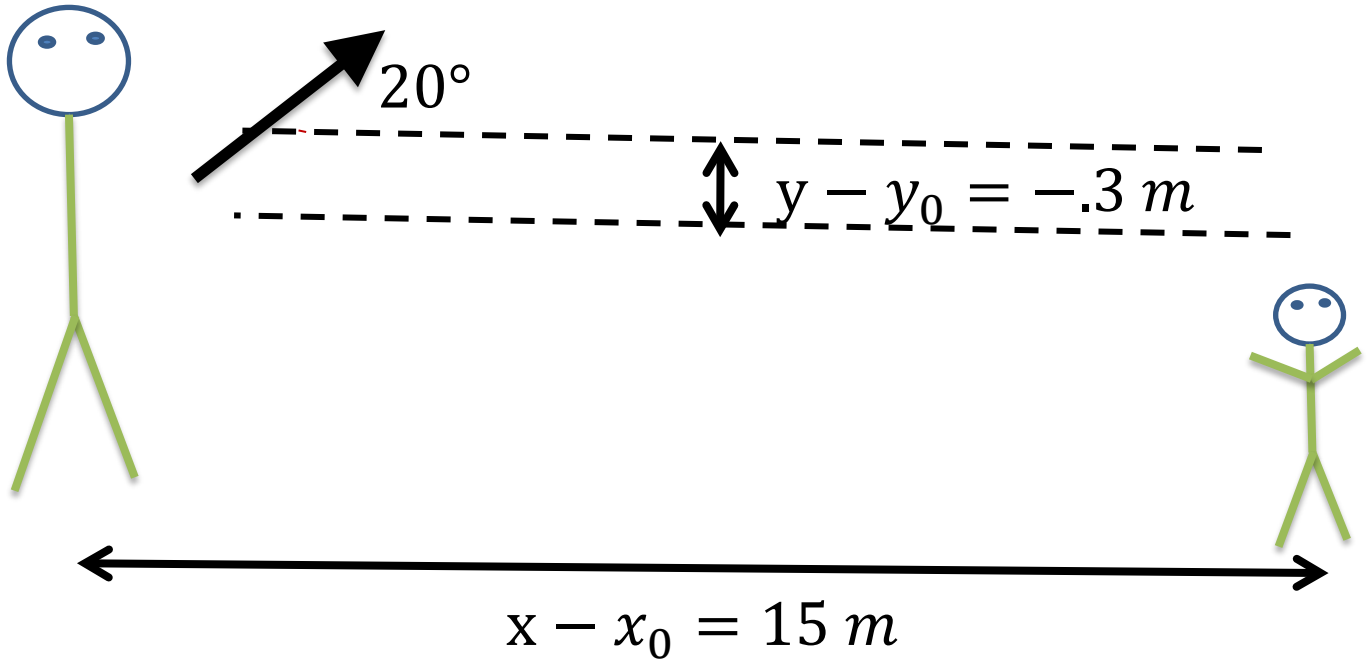
Example 3: William Tell and the apple



Disclaimer: no children (or innocent fruit) were harmed in this example.

Example 3

- William Tell aims arrow at an apple, 15 m away, on his son's head. The apple is .3 m below the initial height of the arrow. (Take $x_0 = y_0 = 0$.) Assume launch angle is $20°$ above the horizon.



Example 3 (continued)

(a) What must initial launch speed, v_0 , be?

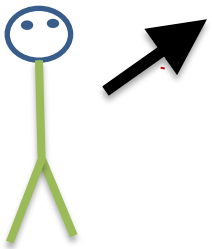
Recall the parabola equation for the trajectory:

$$y = (\tan\theta)x - \left(\frac{g}{2v_0^2 cos^2\theta}\right)x^2$$

Solve for initial speed:

$$v_0 = \sqrt{\frac{gx^2}{2cos^2\theta(x\tan\theta - y)}} = \sqrt{\frac{(9.8)(15)^2}{2cos^2 20°((15)\tan 20° - (-.3))}}$$

$$v_0 = \boxed{14.72 \text{ m/s}}$$



Given: $y - y_0 = -.3\ m$

$x - x_0 = 15\ m$

$\theta = 20°$

Example 3 (continued)

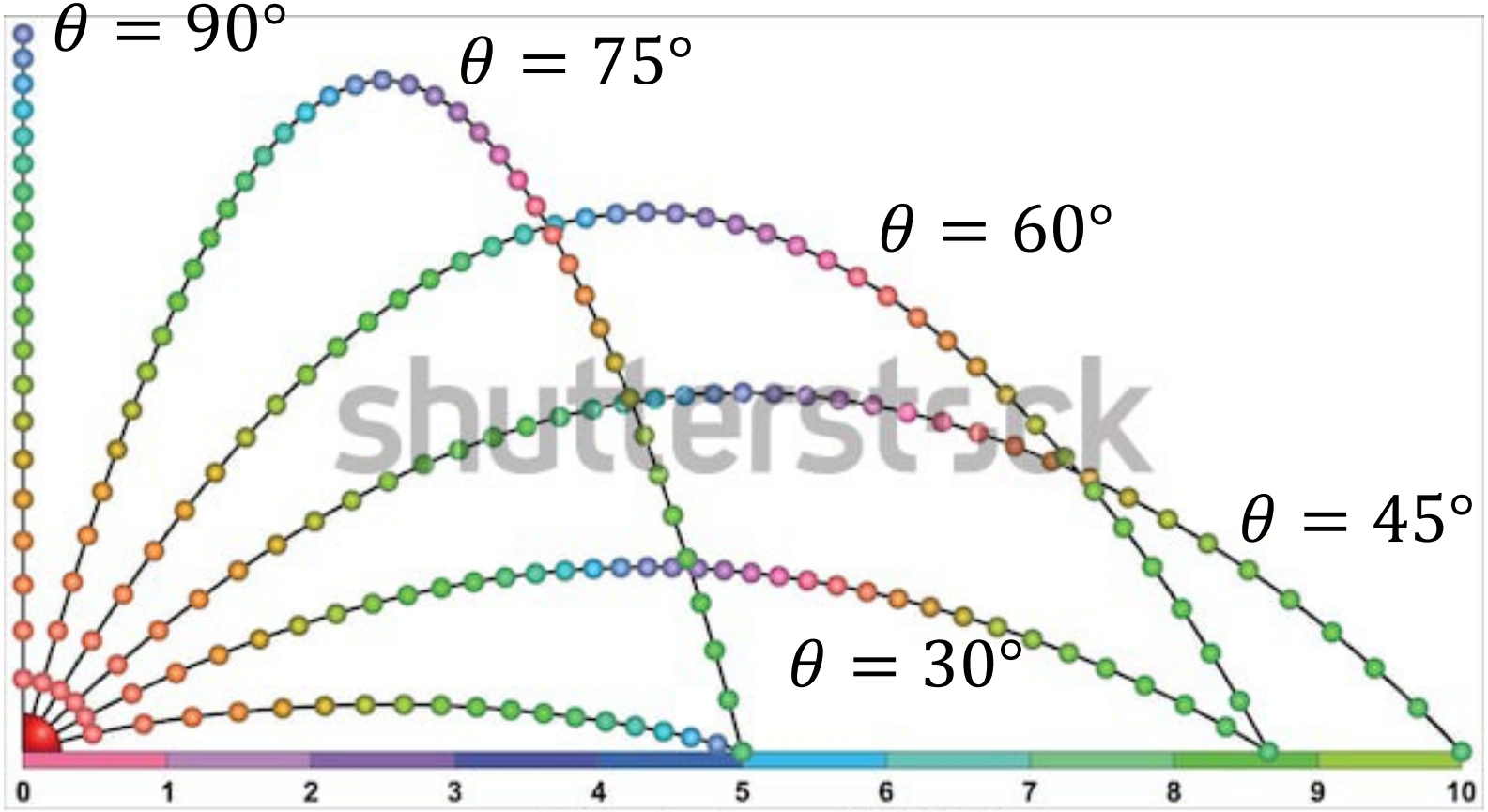
(b) How long is the arrow in the air?

$$x = v_{0x}t \quad \rightarrow \quad t = \frac{x}{v_{0x}} = \frac{x}{v_0 cos\theta} = \frac{15}{(14.72)\cos 20°}$$

$$t = \boxed{1.08\ s}$$

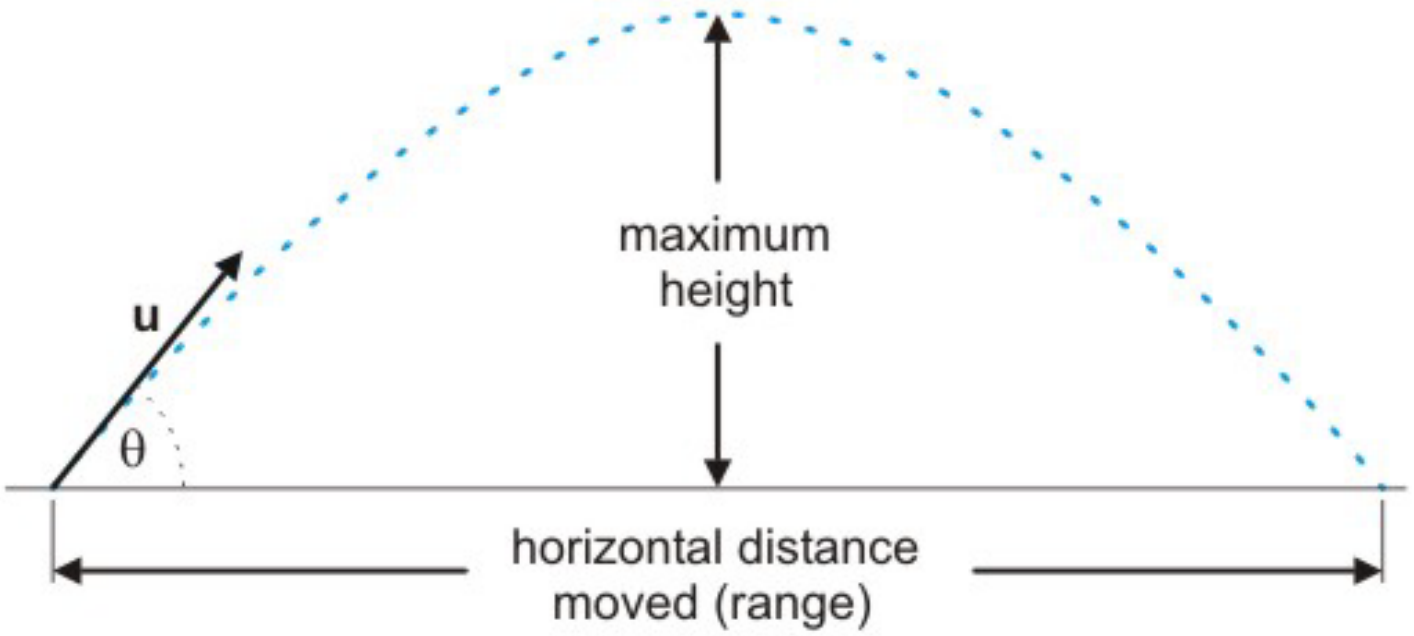
Height and distance

- For fixed launch speed, both height and distance traveled depend on launch angle



Range

- The **<u>range</u> R** is the distance a projectile travels horizontally before returning to its initial height.



Range

- To find R, set $y = y_0$ and solve for $R = x - x_0$.

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \xrightarrow{y = y_0} 0 = v_{0y}t - \frac{1}{2}gt^2$$

- Solve for t:

Launch point

Landing point

$$0 = \left(v_{0y} - \frac{1}{2}gt\right)t \quad \rightarrow \quad t = 0 \quad or \quad t = \frac{2v_{0y}}{g}$$

- Use t to find R:

$$R = x - x_0 = v_{0x}t = v_{0x}\left(\frac{2v_{0y}}{g}\right) = \left(\frac{2(v_0\cos\theta)(v_0\sin\theta)}{g}\right)$$

(Using $v_{0x} = v_0\cos\theta$ and $v_{0y} = v_0\sin\theta$)

- We can simplify this result

Range

- We have: $$R = \left(\frac{2v_0^2 \cos\theta \ \sin\theta}{g} \right)$$

- But recall the trig identity: $\sin 2\theta = 2 \sin\theta \cos\theta$
- So: $$\boxed{R = \left(\frac{v_0^2 \sin 2\theta}{g} \right)}$$

- **Dire Warning: Do not use this formula unless initial and final heights are the same.**

Maximum Range

For given v_0, maximum range occurs at $\theta = 45°$

- **Proof 1:** The range is given by $R = \left(\frac{v_0^2 \sin 2\theta}{g}\right)$. But v_0 and g are fixed, so R is maximum when $\sin 2\theta$ is maximum. The maximum of the sine function occurs when $2\theta = 90°$ ($\sin 90° = 1$), so: $\theta = 45°$
- **Proof 2:** Maximum occurs when $\frac{dR}{d\theta} = 0$. So:

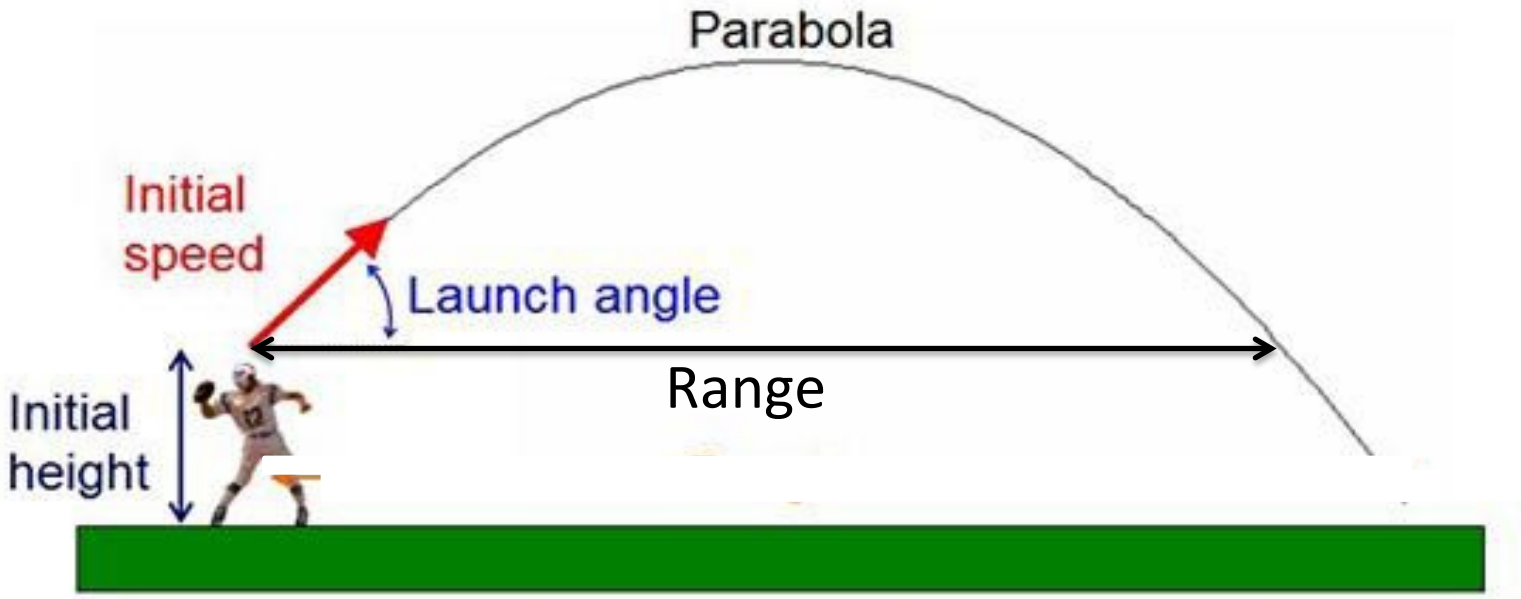
$$0 = \frac{dR}{d\theta} = \frac{v_0^2}{g}\frac{d}{d\theta}(\sin 2\theta) = \frac{v_0^2}{g}(-2\cos 2\theta) = -\frac{2v_0^2}{g}\cos 2\theta$$

Maximum occurs when:

$$\cos 2\theta = 0 \rightarrow 2\theta = 90° \rightarrow \theta = 45°$$

Example 4

- A quarterback throws the ball from 35 yards (32 m) the opponent's goal line. Initial speed of the ball is 20 m/s at a 30° launch angle. The ball leaves his hand at an initial height of 2m.



Example 4 (continued)

- Given: $v_0 = 20\frac{m}{s},\ \theta = 30°, y_0 = 2\ m,\ x_0 = 0$

(a) How long does it take for the ball to reach the goal line at $x = 32m$?

$$v_{0x} = v_0 \cos\theta = \left(20\frac{m}{s}\right)\cos 30° = 17.32\ m/s$$

$$v_{0y} = v_0 \sin\theta = \left(20\frac{m}{s}\right)\sin 30° = 10\ m/s$$

$$x = v_{0x}t \quad \rightarrow \quad t = \frac{x}{v_{0x}} = \frac{32m}{17.32m/s} = \boxed{1.85\ s}$$

(b) At what height does the ball cross the goal line?

$$y = y_0 + v_{y0}t - \frac{1}{2}gt^2 = 2 + (10)(1.85) - \frac{1}{2}(9.8)(1.85)^2$$

$$= \boxed{3.73\ m}$$

Continued

Example 4 (continued)

(c) What is the range? $\left(v_0 = 20\frac{m}{s}, \theta = 30°\right)$

$$R = \left(\frac{v_0^2 \sin 2\theta}{g}\right) = \frac{(20\frac{m}{s})^2 \sin 60°}{9.8\ m/s^2} = \boxed{35.35\ m}$$

(d) At what angle must the football be thrown in order to reach receiver 40 m away, i.e. $R = 40\ m$? (Assume v_0 is unchanged and ball is caught at height of 2m.)

$$R = \left(\frac{v_0^2 \sin 2\theta}{g}\right) \quad \rightarrow \quad \sin 2\theta = \frac{gR}{v_0^2} \quad \rightarrow \quad 2\theta = \sin^{-1}\left(\frac{gR}{v_0^2}\right)$$

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{gR}{v_0^2}\right) = \frac{1}{2}\sin^{-1}\left(\frac{(9.8)(40)}{(20)^2}\right) = \frac{1}{2}(78.52°)$$

$$= \boxed{39.26°}$$

Continued

Example 4 (continued)

(e) What is the maximum possible range, assuming v_0 remains held constant at $v_0 = 20\,\frac{m}{s}$?

If is held fixed, the only thing that can be varied to affect the range is the angle. The maximum range occurs at $45°$, so:

$$R_{max} = \left(\frac{v_0^2 \sin 2\theta}{g}\right) = \frac{(20\frac{m}{s})^2 \sin 2(45°)}{9.8\ m/s^2} = \frac{(20\frac{m}{s})^2 \sin(90°)}{9.8\ m/s^2}$$

$$= \boxed{40.82\ m}$$

<u>Note again:</u> the use of the range formula is allowed **<u>only</u>** because we know the initial and final heights are equal.